A replica approach to products of random matrices

1998 J. Phys. A: Math. Gen. 31 951

(http://iopscience.iop.org/0305-4470/31/3/008)

J. Phys. A: Math. Gen. **31** (1998) 951–961. Printed in the UK PII: S0305-4470(98)87019-4

A replica approach to products of random matrices

M Weigt†

Institut für Theoretische Physik, Otto–von-Guericke-Universität Magdeburg PSF 4120, 39016 Magdeburg, Germany

Received 21 August 1997, in final form 22 October 1997

Abstract. We analyse products of random $R \times R$ matrices by means of a variant of the replica trick which was recently introduced for one-dimensional disordered Ising models. The replicated transfer matrix can be block-diagonalized with the help of irreducible representations of the permutation group. We show that the free energy (or the Lyapunov exponent) of the product corresponds to the replica-symmetric representation, whereas replica-asymmetric representations correspond to certain correlation functions.

1. Introduction

The asymptotic properties of products of random matrices play an important role in many physical problems [1, 2]. In models such as disordered one-dimensional magnetic systems they describe the thermodynamic quantities such as free energy or correlations, for localization of electronic waves in random potentials they are related to the transport properties, see also [3]. Such products also appear in the context of chaotic dynamical systems characterizing the divergence of neighbouring trajectories.

Although there are many known results on products of random matrices, some of them even mathematically rigorous, we wish to present a general replica transfer matrix method. Replicas are known to be a very powerful but nevertheless somewhat mysterious tool in the statistical mechanics of disordered systems and related problems [4]. In the case of mean-field models replicas predict the concept of replica symmetry breaking which is related to a highly nontrivial ultrametric structure of states in the low-temperature phase.

The existence and structure of replica-symmetry breaking in low-dimensional systems is not yet clear, see e.g. the argumentation of [5]. In [6] a replica approach to one-dimensional disordered Ising models was presented. Although there does not exist any phase transition at nonzero temperature, a rich replica structure could be observed leading to a 'natural' criterion for replica-symmetry breaking in this special system which is not related to Parisi's replica-symmetry breaking scheme for mean-field models. This criterion is based on the representation structure of the permutation group and could be deduced to a large extent with rigorous methods. In this paper we generalize the approach from disordered 2×2 transfer matrices to infinite products of random $R \times R$ matrices with arbitrary positive integer R. We mainly use the notation of statistical mechanics, i.e. the random matrices are considered as transfer matrices of one-dimensional models with discrete degrees of freedom having R possible values per site and random short-range interactions. We show that the

† E-mail address: martin.weigt@physik.uni-magdeburg.de

representation theoretic approach to replica-symmetry breaking is quite general and can be formulated without specifying a particular one-dimensional model.

Although no replica-symmetry breaking is to be expected for finite R at nonzero temperature, this could be different for $R \to \infty$. One example is the $(1+1)$-dimensional directed polymer in a random medium, for a recent review see [7]. A 'weak' replica-symmetry breaking is expected there [8, 9], but its structure is not yet understood in terms of the conventional Parisi scheme. Our calculations offer a completely different approach to this question. At least for a positive integer replica number n our results are rigorous, and their continuation to $n \to 0$ is straightforward. We therefore hope that the general discussion of this paper offers a new method allowing us to analyse a broad range of interesting one-, $(1+1)$- and two-dimensional models.

This paper is organized as follows. In section 2 we introduce the replicated transfer matrix. For the analysis we need several tools from the representation theory of the symmetric group, these are presented in section 3. In section 4 the replica-symmetric representation space is considered and the free energy is calculated. The connection between replica-asymmetric representations and connected two-point correlation functions is analysed in section 5. It will provide a natural criterion for replica symmetry breaking. In the last section we give a summary and outlook. The paper closes with several appendices containing longer calculations or proofs.

2. The replicated transfer matrix

We consider N $R \times R$ matrices $T^{(i)}, i = 1, \ldots, N$, drawn from a single probability distribution $P(T)$, where R is any positive integer. In the case of a one-dimensional model with random Hamiltonian $H = \sum_i H_i(s_i, s_{i+1})$ (s_i can take R different values) and inverse temperature β they are given by $T^{(i)} = (\exp\{-\beta H_i(s_i, s_{i+1})\})$. For a general distribution these matrices do not commute. Therefore we cannot find a common system of eigenvectors. In order to calculate self-averaging quantities such as the free energy we introduce as usual the n-fold replicated and disorder-averaged partition function,

$$\begin{aligned} \langle\langle Z^n \rangle\rangle &= \left\langle\!\!\left\langle \left(\mathrm{tr} \prod_{i=1}^{N} T^{(i)} \right)^n \right\rangle\!\!\right\rangle \\ &= \mathrm{tr}(\langle\langle T^{\otimes n} \rangle\rangle)^N \end{aligned} \tag{1}$$

where $\langle\langle \cdot \rangle\rangle$ denotes the average with respect to $P(T)$ and $\otimes$ is the Kronecker product of matrices. With this relation we are able to replace the product of N random $R \times R$ matrices by the Nth power of a single $R^n \times R^n$ matrix which can be analysed using standard transfer matrix techniques: we have to find expressions for the eigenvalues of $T_n := \langle\langle T^{\otimes n} \rangle\rangle$ which enable an analytic continuation in n. The free energy is then given by

$$f = -\frac{1}{\beta} \langle\langle \ln Z \rangle\rangle = -\frac{1}{\beta} \lim_{n \to 0} \partial_n \langle\langle Z^n \rangle\rangle \tag{2}$$

which is dominated by the largest eigenvalue of T_n for $n \to 0$. Several correlation lengths can be described by smaller eigenvalues of the same matrix.

To calculate these quantities, some notations will be introduced. The original matrices $T^{(i)}$ act on a R-dimensional vector space V. As a basis we chose any orthonormalized set of R vectors and denote these by $|s\rangle, s = 1, \ldots, R$. Consequently T_n is a linear operator defined on the n-fold tensor product $V^{\otimes n}$ of V with itself which has dimension R^n. The orthonormalized basis vectors of this space are chosen naturally as $|s^1\rangle \otimes |s^2\rangle \otimes \cdots \otimes |s^n\rangle$

$=: |s^1 s^2 \dots s^n\rangle$ where $s^a \in \{1, \dots, R\}$ for all $a = 1, \dots, n$. The matrix elements of T_n are then given by

$$\langle s^1 s^2 \dots s^n | T_n | s'^1 s'^2 \dots s'^n \rangle = \left\langle\!\!\left\langle \prod_{a=1}^{n} \langle s^a | T | s'^a \rangle \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle \prod_{a=1}^{n} T_{s_a, s'_a} \right\rangle\!\!\right\rangle \tag{3}$$

for any two basis vectors of $V^{\otimes n}$.

The average over $P(T)$ produces interactions between the replicas. Nevertheless the replicas are completely equivalent, a renumbering does not change the matrix T_n. This leads to a symmetry of the transfer matrix under replica permutations, i.e. to replica symmetry of T_n. The action of any permutations is given by the R^n-dimensional representation D of the symmetric group $\mathcal{S}_n$:

$$D(\pi)|s^1 s^2 \dots s^n\rangle = |s^{\pi(1)} s^{\pi(2)} \dots s^{\pi(n)}\rangle \qquad \forall \pi \in \mathcal{S}_n \tag{4}$$

whose operator product with T_n commutes,

$$D(\pi) T_n = T_n D(\pi) \qquad \forall \pi \in \mathcal{S}_n. \tag{5}$$

A direct consequence of equation (5) is the closure of any eigenspace of T_n under permutations, these eigenspaces define a subrepresentations of D which in the most general case are irreducible. Further reducibilities would be a hint to a further hidden symmetry.

Consider an element Y of the group algebra $\mathbf{s}_n$ of $\mathcal{S}_n$, i.e. Y is a linear combination of permutations $\pi \in \mathcal{S}_n$. Due to (5) and the linearity of the action of the transfer matrix on $V^{\otimes n}$ it also commutes with T_n. The space $U = Y\, V^{\otimes n} = \sum_{s^1,\dots,s^n} \mathbb{R} Y |s^1 s^2 \dots s^n\rangle$ is therefore invariant under the action of T_n. If we are able to construct elements of $\mathbf{s}_n$ projecting $V^{\otimes n}$ to a proper subspace we can thus achieve a block diagonalization of T_n by its restriction to U and its orthogonal complement $(\mathbf{1} - Y)\, V^{\otimes n}$.

3. Remarks on the symmetric group

In this section we review some properties of the symmetric group and its irreducible representations. These are well studied and numerous excellent presentations can be found, e.g. in [10, 11]. Here we omit any proofs.

The symmetric group $\mathcal{S}_n$ contains the $n!$ permutations of n distinguishable objects. Consider any representation $\tilde{D}$ on a linear space $\tilde{V}$. $\tilde{D}$ is said to be irreducible iff there are no proper subspaces ($\neq \{0\}$) of $\tilde{V}$ closed under $\tilde{D}(\mathcal{S}_n)$. A representation is said to be completely reducible iff it can be decomposed into a direct sum of irreducible subrepresentations. This decomposition is unique up to isomorphisms. Our D defined in the previous section is completely reducible.

The irreducible representations of $\mathcal{S}_n$ are classified by the so-called *standard Young tableaux*. Each Young tableau is characterized by a partition of n, i.e. a set of integers $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m > 0, m \leqslant n$, fulfilling $\sum_a \lambda_a = n$. One arranges m rows of length $\lambda_1, \dots, \lambda_m$ as shown in the figure and fills the boxes with the integers $1, \dots, n$. The tableau is called standard iff the entries of the boxes are increasing within every row and within every column, see e.g. figure 1.

At first we define the row symmetrizer $\mathrm{SYM}_{[\lambda_1,\dots,\lambda_m]} = \prod_{a=1}^{m} \mathrm{SYM}_a$ with SYM_a being the sum of all permutations within the ath row. Then we still need the column antisymmetrizer $\mathrm{ASYM}_{[\lambda_1,\dots,\lambda_m]} = \prod_{b=1}^{\lambda_1} \mathrm{ASYM}_b$ with ASYM_b being the total

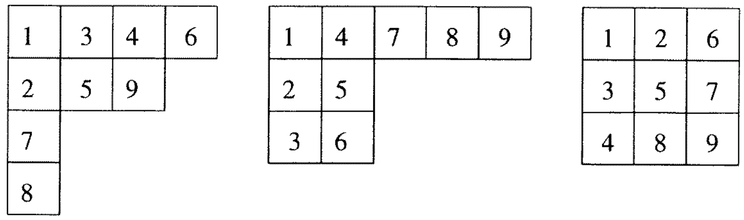

Figure 1. Examples for standard Young tableaux for $n = 9$.

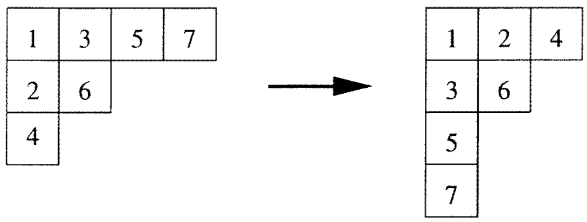

Figure 2. Example for transposing a standard Young tableau.

antisymmetrizer of column b, i.e. the sum of $(-1)^\pi \pi$ over all permutations of this column. $(-1)^\pi$ signifies whether π is odd or even. The *Young operator* is then defined by

$$Y_{[\lambda_1,\ldots,\lambda_m]} = \mathrm{ASYM}_{[\lambda_1,\ldots,\lambda_m]}\,\mathrm{SYM}_{[\lambda_1,\ldots,\lambda_m]} \tag{6}$$

and is an element of the group algebra $\boldsymbol{s}_n$.

If we go back to the representation $\tilde{D}(\mathcal{S}_n)$, then the action of $Y_{[\lambda_1,\ldots,\lambda_m]}$ on any element $|v\rangle$ of $\tilde{V}$ maps this vector to an irreducible subrepresentation. A basis of the irreducible representation space can be constructed by applying all permutations to $Y_{[\lambda_1,\ldots,\lambda_m]}|v\rangle$ and selecting a maximal linearly independent subset. Every standard Young tableau gives a different irreducible representation, those corresponding to the same partition $[\lambda_1, \ldots, \lambda_m]$ but different entries are isomorphic. Depending on the structure of $\tilde{D}$, also the action of the same Young operator on different vectors from $\tilde{V}$ can give different irreducible subrepresentations of $\tilde{D}$. Every irreducible subrepresentation can be constructed in the prescribed way.

Another notion needed in the following is that of the *associate* representation. For any irreducible representation given by a standard Young tableau with partition $[\lambda_1, \ldots, \lambda_n]$ it is given by the *transposed* standard Young tableau, i.e. the rows become the columns and vice versa. The transposed partition is denoted by $[\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{\tilde{m}}]$ with $\tilde{\lambda}_1 = m$ and $\tilde{m} = \lambda_1$. An example is given in the figure 2.

4. The replica-symmetric eigenspaces and free energy

We now return to the problem of finding the eigenvalues of the replicated and disorder-averaged transfer matrix T_n. At the end of section 2 we showed that the space $Y\,V^{\otimes n}$ is invariant with respect to T_n for every $Y \in \boldsymbol{s}_n$. In particular, this is the case for the Young operators which define minimal invariant sets obtainable without further knowledge of the exact form of T_n.

In this section we concentrate on a special irreducible subrepresentation described by the standard Young tableau with only one row. The Young operator $Y_{[n]}$ becomes the symmetrizer of the complete symmetric group, the elements of its image $Y_{[n]}V^{\otimes n}$ are therefore invariant under permutations. The corresponding irreducible subrepresentations of D are thus one-dimensional, all permutations are represented trivially by the identity. Consequently the elements of $Y_{[n]}V^{\otimes n}$ are *replica symmetric* and therefore also the eigenvectors of T_n which are constructed within this space.

As basis vectors for $Y_{[n]}V^{\otimes n}$ we introduce

$$\begin{aligned}|\rho_1,\ldots,\rho_{R-1}\rangle &= \frac{1}{\rho_1!\cdot\ldots\cdot\rho_r!}Y_{[n]}|1\rangle^{\otimes\rho_1}\otimes\cdots\otimes|R\rangle^{\otimes\rho_R}\\ &= \sum_{\{s^a\,|\,\sum_a\delta_{s^a,s}=\rho_s\forall s=1,\ldots,R-1\}}|s^1\ldots s^n\rangle\end{aligned}\tag{7}$$

where ρ_s, $s=1,\ldots,R-1$, and $\rho_R = n-\sum_{s=1}^{R-1}\rho_s$ have to be nonnegative integers. The replica-symmetric submatrix of T_n can be calculated by

$$T_n^{[n]}(\rho_1,\ldots,\rho_{R-1}|\sigma_1,\ldots,\sigma_{R-1}) = \frac{\langle\rho_1,\ldots,\rho_{R-1}|T_n|\sigma_1,\ldots,\sigma_{R-1}\rangle}{\langle\rho_1,\ldots,\rho_{R-1}|\rho_1,\ldots,\rho_{R-1}\rangle}.\tag{8}$$

The denominator results from the fact that the vectors (7) are orthogonal but not normalized. In order to send the replica number n to zero we have to introduce generating functions into the eigenvalue equation

$$\Lambda_{[n]}Z(\sigma_1,\ldots,\sigma_{R-1}) = \sum_{\{\rho_1,\ldots,\rho_{R-1}\}}T_n^{[n]}(\rho_1,\ldots,\rho_{R-1}|\sigma_1,\ldots,\sigma_{R-1})Z(\rho_1,\ldots,\rho_{R-1})\tag{9}$$

by writing

$$\Phi[x_1,\ldots,x_{R-1}] = \sum_{\{\rho_1,\ldots,\rho_{R-1}\}}x_1^{\rho_1}\cdot\ldots\cdot x_{R-1}^{\rho_{R-1}}Z(\rho_1,\ldots,\rho_{R-1}).\tag{10}$$

The eigenvalue equation (9) now reads

$$\Lambda_{[n]}\Phi[x_1,\ldots,x_{R-1}] = \left\langle\!\!\left\langle\left(\sum_{s=1}^{R}x_sT_{R,s}\right)^n\cdot\Phi\left[\frac{\sum_s x_sT_{1,s}}{\sum_s x_sT_{R,s}},\ldots,\frac{\sum_s x_sT_{R-1,s}}{\sum_s x_sT_{R,s}}\right]\right\rangle\!\!\right\rangle\tag{11}$$

where we introduced $x_R = 1$ for simplicity, for the calculations see appendix A. In this equation a sensible limit $n\to 0$ can be performed. The largest eigenvalue is $\Lambda = 1-\beta nf+\mathrm{O}(n^2)$ where β is the inverse temperature and f the replica-symmetric free energy. Finally, we change from left to right eigenfunctions, introduce $\boldsymbol{x}=(x_1,\ldots,x_{R-1})\in\mathbb{R}^{R-1}$ and

$$h_r(\boldsymbol{x}) = \frac{\sum_{s=1}^{R}x_sT_{r,s}}{\sum_{s=1}^{R}x_sT_{R,s}}\tag{12}$$

and obtain an equation for a $(R-1)$-dimensional invariant density

$$\Phi^{(0)}[\boldsymbol{x}] = \int \mathrm{d}^{R-1}y\,\langle\!\langle\delta^{(R-1)}(\boldsymbol{x}-\boldsymbol{h}(\boldsymbol{y}))\rangle\!\rangle\Phi^{(0)}[\boldsymbol{y}]\tag{13}$$

where we used the $(R-1)$-dimensional Dirac distribution $\delta^{(R-1)}(\cdot)$. The density has to be normalized, $\int\mathrm{d}^{(R-1)}x\,\Phi^{(0)}[\boldsymbol{x}]=1$†. As in perturbation theory we calculate the $\mathrm{O}(n)$-corrections of λ with the unperturbated eigenfunction,

$$f = -\frac{1}{\beta}\int\mathrm{d}^{(R-1)}x\,\Phi^{(0)}[\boldsymbol{x}]\left\langle\!\!\left\langle\ln\left(\sum_{s=1}^{R}x_sT_{R,s}\right)\right\rangle\!\!\right\rangle.\tag{14}$$

† This last point still remains somewhat mysterious—by changing from left to right eigenfunctions we also change the function space from polynomials to functions having a finite integral. Up until now this step is only justified by its results (13), (14) and there coincidence with the results of [12].

In this paper we do not calculate this free energy for any special distribution. This task itself is very hard and has been solved only for a few distributions of quenched disorder, see e.g. [1, 2] and references therein.

The same equations can also be obtained without using replicas. For the one-dimensional Ising model this was established by Derrida and Hilhorst [12], their method using Riccati variables also generalizes to more complicated degrees of freedom than Ising spins. We will sketch this derivation of (13) and (14) shortly. Consider again the original disordered transfer matrices $T^{(i)}$, $i = 1, \ldots, N$. They define the R-dimensional iteration relation

$$Z_r^{(i+1)} := \sum_{s=1}^{R} T_{r,s}^{(i)} Z_s^{(i)} \tag{15}$$

from which the partition function can be calculated as $Z = \sum_r Z_r^{(N)}$. We introduce Riccati variables $x_s^{(i)} = Z_s^{(i)}/Z_R^{(i)}$ following the iteration relation

$$x_r^{(i+1)} = \frac{\sum_{s=1}^{R} x_s^{(i)} T_{r,s}^{(i)}}{\sum_{s=1}^{R} x_s^{(i)} T_{R,s}^{(i)}} \tag{16}$$

cf (12). Thus, in the limit of large i these random vectors can be described by the invariant density (13). The free energy per site can be calculated from the asymptotic properties of $-\frac{1}{\beta}\ln(Z_R^{(i+1)}/Z_R^{(i)})$, it is therefore given by (14). We consequently conclude that the free energy is always given correctly by its replica-symmetric value.

Another result reminiscent of ours was obtained by Lin [13], who showed the equivalence of an early replica approach by Kac with Dyson's method for the phonon spectrum of a chain of random masses and springs.

5. Replica-asymmetric eigenspaces and two-point correlations

From the example of the (1+1)-dimensional polymer in a random medium [7] we know that the replica-symmetric free energy and scaling laws can be correct, but the replica-symmetry is nevertheless weakly broken. In order to establish an internal check if this is the case or not, we have to consider the eigenvalues of the other—'replica symmetry broken'—subrepresentations, and we have to compare them with the largest replica-symmetric eigenvalue. If we find any degeneracy between these for $n < 0$, we expect a replica-symmetry breaking of a possibly new type. The asymmetric eigenvalues will be found to have a very simple interpretation in terms of connected two-point correlation functions supporting this statement.

At first we have to clarify which irreducible representations of $\mathcal{S}_n$ are subrepresentations of D. They are characterized by a nonempty representation space $Y_{[\lambda_1,\ldots,\lambda_m]}V^{\otimes n}$. Due to the column antisymmetrization in the definition of the Young operator this is only the case if the number m of rows in the corresponding standard Young tableau does not exceed the dimension R of the linear space V. So we can restrict our considerations to $m < R$.

Consider the operator

$$X|s\rangle = x_s|s\rangle \tag{17}$$

where x_s is any observable assigned to the basis vectors $|s\rangle$, e.g. spin, location, or occupation number. It can be simply extended to the replicated vector space $V^{\otimes n}$ by introducing the n operators $X_a^{(n)} = \mathbf{1}^{\otimes a-1} \otimes X \otimes \mathbf{1}^{\otimes n-a}$, $a = 1, \ldots, n$. They are commutative and measure the value of x at the ath replica site. In addition we introduce the operators

$$X^{(\lambda)} = \begin{cases} \mathbf{1} & \text{if } \lambda = 1 \\ \prod_{1\leqslant a<b\leqslant\lambda} (X_a^{(\lambda)} - X_b^{(\lambda)}) & \text{if } \lambda > 1 \end{cases} \tag{18}$$

for any nonnegative integer λ. For every partition $[\lambda_1, \dots, \lambda_m]$ and its transpose $[\tilde{\lambda_1}, \dots, \tilde{\lambda}_{\tilde{m}}]$, they can be combined to the operator

$$X_{[\lambda_1,\dots,\lambda_m]} = X^{(\tilde{\lambda_1})} \otimes \cdots \otimes X^{(\tilde{\lambda}_{\tilde{m}})} \tag{19}$$

acting on $V^{\otimes n}$. Moreover, it maps any replica-symmetric vector to a vector in a representation space belonging to the standard Young tableau with m rows of length $\lambda_1, \dots, \lambda_m$ where we fill one column after another successively with integers $1, \dots, n$. An example is given by the second tableau in figure 1. A sketch of the proof will be shown in appendix B.

Using this we find that

$$\mathrm{tr}(T_n^i X_{[\lambda_1,\dots,\lambda_m]} T_n^{j-i} X_{[\lambda_1,\dots,\lambda_m]} T_n^{N-j}) \propto \Lambda_{[\lambda_1,\dots,\lambda_m]}^{|j-i|} \cdot \Lambda_{[n]}^{N-|j-i|} \tag{20}$$

for large distances $|j - i|$. $\Lambda_{[\lambda_1,\dots,\lambda_m]}$ is the largest eigenvalue if we consider only eigenfunctions in the subspace $Y_{[\lambda_1,\dots,\lambda_m]}V^{\otimes n}$.

Because of $\sum_a \lambda_a = n$ we have by definition (19)

$$X_{[\lambda_1,\dots,\lambda_m]} = X_{[\lambda_2,\lambda_2,\dots,\lambda_m]} \otimes \mathbf{1}^{n-\sum_{a=2}^m \lambda_a - \lambda_2}. \tag{21}$$

Introducing this into (20) we can send $n \to 0$ and obtain

$$\langle\langle\langle X_{[\lambda_2,\lambda_2,\dots,\lambda_m]}(i) \cdot X_{[\lambda_2,\lambda_2,\dots,\lambda_m]}(j)\rangle\rangle\rangle \propto \lim_{n\to 0} \Lambda_{[n-\sum_{a=2}^m \lambda_a,\lambda_2,\dots,\lambda_m]}^{|j-i|} \tag{22}$$

i.e. the two-point correlation function of $X_{[\lambda_2,\lambda_2,\dots,\lambda_m]}$ decays exponentially with correlation length $\xi = -1/\ln \Lambda_{[-\sum_{a=2}^m \lambda_a,\lambda_2,\dots,\lambda_m]}$. $\langle\cdot\rangle$ denotes the thermodynamic average in the disordered system with transfer matrices $T^{(i)}, i = 1, \dots, N$. In order to calculate this we still need $2 \cdot \sum_{a=2}^m \lambda_a$ real noninteracting replicas of the original quenched system.

Here we concentrate on Young tableaux having only two rows, i.e. to partitions $[n-\lambda, \lambda]$. There the operator reads

$$X_{[n-\lambda,\lambda]} = (X \otimes \mathbf{1} - \mathbf{1} \otimes X)^{\otimes\lambda} \otimes \mathbf{1}^{\otimes(n-2\lambda)} \tag{23}$$

and consequently

$$\langle\langle\langle X_{[\lambda,\lambda]}(i) \cdot X_{[\lambda,\lambda]}(j)\rangle\rangle\rangle = \langle\langle(\langle x_{s_i} x_{s_j}\rangle - \langle x_{s_i}\rangle\langle x_{s_j}\rangle)^\lambda\rangle\rangle \propto \Lambda_{[-\lambda,\lambda]}^{|j-i|} \tag{24}$$

describes the λth moment of the connected two-point correlation function with respect to the disorder distribution. The correlation length diverges whenever $\lim_{n\to 0} \Lambda_{[n-\lambda,\lambda]} = 1$. The criterion for replica-symmetry breaking, i.e. the degeneracy of the largest replica-symmetric eigenvalue with an asymmetric one, thus coincides with the standard criterion for a phase transition. In principle, such a transition can be expected in special systems for zero temperature or in the limit of infinite R.

In appendix C we will develop an equation for $\lim_{n\to 0} \Lambda_{[n-\lambda,\lambda]}$. The calculations are quite similar to the replica-symmetric one, but due to the more complicated representation structure they are somewhat lengthy. Here we give only the final result, an eigenvalue equation for a function

$$\mathbf{\Phi}^{[-\lambda,\lambda]} : \mathbb{R}^{R-1} \to \mathbb{R}^{((R-1)^\lambda)} \tag{25}$$

given by its components $\Phi^{[-\lambda,\lambda]}_{s^1,\ldots,s^\lambda}(\boldsymbol{x})$:

$$\Lambda_{[-\lambda,\lambda]}\Phi^{[-\lambda,\lambda]}_{s^1,\ldots,s^\lambda}(\boldsymbol{x}) = \int \mathrm{d}^{R-1}y \sum_{r^1,\ldots,r^\lambda=1}^{R-1} \left\langle\!\!\left\langle \delta^{(R-1)}(\boldsymbol{x}-\boldsymbol{h}(\boldsymbol{y})) \prod_{a=1}^{\lambda} \frac{\partial h_{s^a}}{\partial y_{r^a}} \right\rangle\!\!\right\rangle \Phi^{[-\lambda,\lambda]}_{r^1,\ldots,r^\lambda}(\boldsymbol{y}). \tag{26}$$

This equation reflects that correlation functions are not self-averaging.

For every eigenfunction $\boldsymbol{\Phi}^{[-1,1]}(\boldsymbol{x})$ of $T_n^{[n-1,1]}$ for $n \to 0$ the function $\nabla\cdot\boldsymbol{\Phi}^{[-1,1]}(\boldsymbol{x})$ is an eigenfunction of the replica-symmetric transfer matrix given in (13) to the same eigenvalue. Only the largest replica-symmetric eigenvalue (= 1) cannot be reached in this way, because the integral of $\nabla\cdot\boldsymbol{\Phi}^{[-1,1]}(\boldsymbol{x})$ over the definition space $\mathbb{R}^{R-1}$ vanishes. Therefore the largest eigenvalue of $T_0^{[-1,1]}$ equals the second largest of $T_0^{[0]}$ etc. The first transfer matrix block which could produce a diverging correlation length outside the replica-symmetric sector is the one corresponding to $[n-2, 2]$, i.e. to the second moment of the connected two-point correlation function. This is know to be correct for several disordered systems, e.g. spin glasses where the second moment of the connected two-point function describes the nonlinear susceptibility [14].

The same procedure can in principle be used in the analysis of the transfer matrix blocks corresponding to Young tableaux having $3, \ldots, R$ rows, but their eigenvalues correspond to very strange correlation functions having no obvious physical significance. So we refrain from doing it.

6. Summary and outlook

In this paper we have developed a general replica transfer matrix method capable of handling products of random finite-dimensional matrices. The obtained replica-symmetric expression for the free energy (or Lyapunov exponent) was found to be correct by comparison with previously known rigorous results. The correlation lengths of the moments of the connected correlation function with respect of the quenched disorder could be evaluated only by taking into account also the replica-asymmetric representations. The divergence of one of these correlation lengths gave a 'natural' criterion for replica symmetry breaking in such systems which is not related to Parisi's hierarchical breaking scheme. So we showed that the representation theoretic approach to the replica structure is a general tool for one-dimensional models and it can be probably extended to two-dimensional models by considering larger and larger one-dimensional stripes. We hope that the application of these general results to particular models can help to understand the replica-symmetry breaking structure in low dimensions.

Acknowledgments

The author is very grateful to J Berg and R Monasson for fruitful discussions and careful reading the manuscript.

Appendix A. Laplace transform of the replica-symmetric eigenvalue equation

In this appendix we calculate the Laplace transform of the replica-symmetric eigenvalue equation (9). We start with

$$\Lambda_{[n]}Z(\sigma_1,\ldots,\sigma_{R-1}) = \sum_{\{\rho_1,\ldots,\rho_{R-1}\}} T_n^{[n]}(\rho_1,\ldots,\rho_{R-1}|\sigma_1,\ldots,\sigma_{R-1})Z(\rho_1,\ldots,\rho_{R-1}) \tag{A1}$$

where the replica-symmetric transfer matrix is given by

$$T_n^{[n]}(\rho_1,\ldots,\rho_{R-1}|\sigma_1,\ldots,\sigma_{R-1}) = \frac{\langle\rho_1,\ldots,\rho_{R-1}|T_n|\sigma_1,\ldots,\sigma_{R-1}\rangle}{\langle\rho_1,\ldots,\rho_{R-1}|\rho_1,\ldots,\rho_{R-1}\rangle} \tag{A2}$$

using the replica-symmetric vectors

$$|\rho_1,\ldots,\rho_{R-1}\rangle = \sum_{\{s^a|\sum_a \delta_{s^a,s}=\rho_s \forall s=1,\ldots,R-1\}} |s^1\ldots s^n\rangle \tag{A3}$$

see section 4. If we introduce the Laplace transformation (10) on the left-hand side of (A1) we obtain (introducing $x_R = 1$, $\rho_R = n - \rho_1 - \cdots - \rho_{R-1}$)

$$\begin{aligned}
\Lambda_{[n]}\Phi[x_1,\ldots,x_{R-1}] &= \sum_{\rho_1,\ldots,\rho_{R-1}} Z(\rho_1,\ldots,\rho_{R-1}) \\
&\quad\times \sum_{\sigma_1,\ldots,\sigma_{R-1}} x_1^{\sigma_1}\cdot\ldots\cdot x_{R-1}^{\sigma_{R-1}} T_n^{[n]}(\rho_1,\ldots,\rho_{R-1}|\sigma_1,\ldots,\sigma_{R-1}) \\
&= \sum_{\rho_1,\ldots,\rho_{R-1}} Z(\rho_1,\ldots,\rho_{R-1}) \sum_{\{s^a\}} x_1^{\Sigma_a\delta_{s^a,1}}\cdot\ldots\cdot x_{R-1}^{\Sigma_a\delta_{s^a,R-1}} \langle 1|^{\otimes\rho_1} \\
&\quad\otimes\cdots\otimes\langle R|^{\otimes\rho_R} T_n|s^1\ldots s^n\rangle \\
&= \sum_{\rho_1,\ldots,\rho_{R-1}} Z(\rho_1,\ldots,\rho_{R-1}) \left(\sum_{s=1}^R x_s T_{1,s}\right)^{\rho_1}\cdot\ldots\cdot\left(\sum_{s=1}^R x_s T_{R,s}\right)^{\rho_R} \\
&= \left(\sum_{s=1}^R x_s T_{R,s}\right)^n \sum_{\rho_1,\ldots,\rho_{R-1}} Z(\rho_1,\ldots,\rho_{R-1}) \prod_{r=1}^{R-1}\left(\frac{\sum_{s=1}^R x_s T_{r,s}}{\sum_{s=1}^R x_s T_{R,s}}\right)^{\rho_r} \\
&= \left(\sum_{s=1}^R x_s T_{R,s}\right)^n \cdot \Phi\left[\frac{\sum_s x_s T_{1,s}}{\sum_s x_s T_{R,s}},\ldots,\frac{\sum_s x_s T_{R-1,s}}{\sum_s x_s T_{R,s}}\right]
\end{aligned} \tag{A4}$$

which is equation (11).

Appendix B. Proof of section 5

In this appendix we show that the operators $X_{[\lambda_1,\ldots,\lambda_m]}$ defined in (19) map any replica-symmetric vector to a representation space for an irreducible representation with a Young tableau described by $[\lambda_1,\ldots,\lambda_m]$. This can be done by proving the equation

$$Y_{[\lambda_1,\ldots,\lambda_m]} X_{[\lambda_1,\ldots,\lambda_m]} Y_{[n]} = c_{[\lambda_1,\ldots,\lambda_m]} X_{[\lambda_1,\ldots,\lambda_m]} Y_{[n]} \tag{B1}$$

where $c_{[\lambda_1,\ldots,\lambda_m]}$ is a real number given by $Y^2_{[\lambda_1,\ldots,\lambda_m]} = c_{[\lambda_1,\ldots,\lambda_m]} Y_{[\lambda_1,\ldots,\lambda_m]}$. Here we concentrate on the case $[n-\lambda,\lambda]$, i.e. to Young tableaux with only two rows. These are the most important cases for our needs, and the proof can be generalized directly to more complicated tableaux as well using analogous procedures.

In the case of two rows we have

$$Y_{[n-\lambda,\lambda]} = (1-(1,2))(1-(3,4))\ldots(1-(2\lambda-1,2\lambda))\cdot\mathrm{SYM}_{[n-\lambda,\lambda]} \tag{B2}$$

where (a,b) denotes the transposition permuting a and b, and

$$X_{[n-\lambda,\lambda]} = (X\otimes\mathbf{1} - \mathbf{1}\otimes X)^{\otimes\lambda}\otimes\mathbf{1}^{\otimes(n-2\lambda)}. \tag{B3}$$

(i) As a first step we note that

$$\begin{aligned}
\forall\pi\in\mathcal{S}_n : \pi X_{a_1}^{(n)}\cdot\ldots\cdot X_{a_l}^{(n)} Y_{[n]} &= X_{\pi(a_1)}^{(n)}\cdot\ldots\cdot X_{\pi(a_l)}^{(n)}\pi Y_{[n]} \\
&= X_{\pi(a_1)}^{(n)}\cdot\ldots\cdot X_{\pi(a_l)}^{(n)} Y_{[n]}.
\end{aligned} \tag{B4}$$

It follows that $\mathrm{SYM}_{[n-\lambda,\lambda]}\, X_{[n-\lambda,\lambda]} Y_{[n]}$ is a sum of certain $X_{a_1}^{(n)} \cdot \ldots \cdot X_{a_\lambda}^{(n)} Y_{[n]}$ with integer prefactors depending on $a_1 < \cdots < a_\lambda$.

(ii) The action of $\mathrm{ASYM}_{[n-\lambda,\lambda]} = (1-(1,2))(1-(3,4))\ldots(1-(2\lambda-1,2\lambda))$ on these gives

$$\begin{aligned}
\mathrm{ASYM}_{[n-\lambda,\lambda]}\, & X_{a_1}^{(n)} \cdot \ldots \cdot X_{a_\lambda}^{(n)} Y_{[n]} \\
&= \begin{cases} \pm X_{[n-\lambda,\lambda]} Y_{[n]} & \text{if } a_\rho \in \{2\rho-1, 2\rho\} \forall \rho = 1, \ldots, \lambda \\ 0 & \text{else.} \end{cases}
\end{aligned} \tag{B5}$$

If there were the factors $X_{2\rho-1}^{(n)}$ and $X_{2\rho}^{(n)}$ for any $\rho \leqslant \lambda$, the action of $(1-(2\rho-1,2\rho))$ would annihilate the term. The same happens, if there is any $\rho \leqslant \lambda$ for which neither $X_{2\rho-1}^{(n)}$ nor $X_{2\rho}^{(n)}$ appear in the product. The sign in (B5) can be obtained by counting the even indices a_ρ in $X_{a_1}^{(n)} \cdot \ldots \cdot X_{a_\lambda}^{(n)}$.

Altogether we find that the action of the Young operator produces only a constant of proportionality, and the proof is complete.

Appendix C. Calculation of asymmetric eigenvalue equations

In this appendix we present the calculation of the eigenvalue equations for nontrivial irreducible representations at the example of $[n-1,1]$. This case is surely the simplest nontrivial one, but the ideas of the calculation are the same also for higher representations.

We consider the standard Young tableau for the partition $[n-1,1]$ having entries $1, 3, 4, \ldots, n$ in the first row and 2 in the second. The corresponding Young operator

$$Y_{[n-1,1]} = (1-(1,2)) \cdot \mathrm{SYM}(1,3,4,\ldots,n) \tag{C1}$$

maps an arbitrary basis vector $|s^1 \ldots s^n\rangle$ up to a normalization constant to

$$\begin{aligned}
|s^2; \sigma_1, \ldots, \sigma_{R-1}\rangle &:= \sum_{s \neq s^2} (|s s^2\rangle - |s^2 s\rangle) \\
&\quad \otimes |\sigma_1, \ldots, \sigma_{\min(s,s^2)} - 1, \ldots, \sigma_{\max(s,s^2)} - 1, \ldots, \sigma_{R-1}\rangle
\end{aligned} \tag{C2}$$

where the last term in the product is a symmetrized vector in the $(n-2)$-fold replicated vector space, cf equation (7), and $\sigma_s = \sum_{a=1}^n \delta_{s^a,s}$. Due to $Y_{[n-1,1]} T_n = T_n Y_{[n-1,1]}$ these vectors form an invariant set with respect to T_n. For given $\sigma_1, \ldots, \sigma_{R-1}$ there are $R-1$ linearly independent vectors of this type, so without loss of generality we can choose $s^2 = 1, \ldots, R-1$.

The transfer matrix block to be calculated is

$$\begin{aligned}
T_n^{[n-1,1]}(s; \sigma_1, \ldots, \sigma_{R-1} | r; \rho_1, \ldots, \rho_{R-1}) &= \frac{\langle s; \sigma_1, \ldots, \sigma_{R-1} | T_n | r; \rho_1, \ldots, \rho_{R-1}\rangle}{\langle s; \sigma_1, \ldots, \sigma_{R-1} | s; \sigma_1, \ldots, \sigma_{R-1}\rangle} \\
&= \sum_{t \neq r} (T_{R,t} T_{s,r} - T_{R,r} T_{s,t}) (T^{\otimes(n-2)})^{[n-2]} \\
&\quad \times (\sigma_1, \ldots, \sigma_s - 1, \ldots, \sigma_{R-1} | \rho_1, \ldots, \rho_r - 1, \ldots, \rho_{R-1}).
\end{aligned} \tag{C3}$$

The matrix $(T^{\otimes(n-2)})^{[n-2]}$ is nothing but the replica symmetric matrix $T_{n-2}^{[n-2]}$ *without* the average over the quenched disorder. For the eigenvalue equation

$$\begin{aligned}
\Lambda_{[n-1,1]} \cdot C(r; \rho_1, \ldots, \rho_{R-1}) &= \sum_{s; \sigma_1, \ldots, \sigma_{R-1}} T_n^{[n-1,1]}(s; \sigma_1, \ldots, \sigma_{R-1} | r; \rho_1, \ldots, \rho_{R-1}) \\
&\quad \times C(s; \sigma_1, \ldots, \sigma_{R-1})
\end{aligned} \tag{C4}$$

we introduce again a Laplace transform by

$$\Phi_s[x_1, \ldots, x_{R-1}] = \sum_{\sigma_1,\ldots,\sigma_{R-1}} x_1^{\sigma_1} \cdot \ldots \cdot x_s^{\sigma_s - 1} \cdot \ldots \cdot x_{R-1}^{\sigma_{R-1}} C(s; \sigma_1, \ldots, \sigma_{R-1}). \tag{C5}$$

Due to ($x_R := 1$)

$$\begin{aligned}
&\sum_{\rho_1,\ldots,\rho_{R-1}} x_1^{\rho_1} \cdot \ldots \cdot x_r^{\rho_r - 1} \cdot \ldots \cdot x_{R-1}^{\rho_{R-1}} T_n^{[n-1,1]}(s; \sigma_1, \ldots, \sigma_{R-1} | r; \rho_1, \ldots, \rho_{R-1}) \\
&\quad = \left\langle\!\!\left\langle \sum_{t \neq r} (T_{R,t} T_{s,r} - T_{R,r} T_{s,t}) x_t \right.\right. \\
&\qquad \times \left.\left. \left(\sum_{\rho_1,\ldots,\rho_{R-1}} x_1^{\rho_1} \cdot \ldots \cdot x_r^{\rho_r - 1} \cdot \ldots \cdot x_t^{\rho_t - 1} \cdot \ldots \cdot x_{R-1}^{\rho_{R-1}} (T^{\otimes(n-2)})^{[n-2]}(\ldots | \ldots) \right) \right\rangle\!\!\right\rangle \\
&\quad = \left\langle\!\!\left\langle \sum_{t \neq r} (T_{R,t} T_{s,r} - T_{R,r} T_{s,t}) x_t \left(\sum_{p=1}^{R} x_p T_{R,p} \right)^{n-2} \prod_{q=1}^{R-1} \left(\frac{\sum_{p=1}^{R} x_p T_{q,p}}{\sum_{p=1}^{R} x_p T_{R,p}} \right)^{\sigma_q - \delta_{q,s}} \right\rangle\!\!\right\rangle \\
&\quad = \left\langle\!\!\left\langle \left(\sum_{p=1}^{R} x_p T_{R,p} \right)^{n} \cdot \frac{\partial h_s(\boldsymbol{x})}{\partial x_r} \cdot \prod_{q=1}^{R-1} h_q(\boldsymbol{x})^{\sigma_q - \delta_{q,s}} \right\rangle\!\!\right\rangle
\end{aligned} \tag{C6}$$

where the second last step is the same as in appendix A for the replica-symmetric case, and where we use the function $\boldsymbol{h}(\boldsymbol{x})$ defined in (12), the eigenvalue equation becomes

$$\Lambda_{[n-1,1]} \Phi_r[x_1, \ldots, x_{R-1}] = \left(\sum_{p=1}^{R} x_p T_{R,p} \right)^{n} \sum_{s=1}^{R-1} \frac{\partial h_s}{\partial x_r} \Phi_s[h_1(\boldsymbol{x}), \ldots, h_{R-1}(\boldsymbol{x})]. \tag{C7}$$

In the limit $n \to 0$ this results in equation (26) for $\lambda = 1$. The calculations for larger λ are analogous.

References

[1] Luck J M 1992 *Systèmes désordonnés Unidimensionels* (Saclay: Aléa)
[2] Crisanti A, Paladin G and Vulpiani A 1993 *Products of Random Matrices in Statistical Physics* (Berlin: Springer)
[3] Pendry J B 1994 *Adv. Phys.* **43** 461
[4] Mézard M, Parisi G and Virasoro M A 1987 *Spin Glass Theory and Beyond* (Singapore: World Scientific)
[5] Newman C M and Stein D L 1998 *Phys. Rev.* E **57** to appear
[6] Weigt M and Monasson R 1996 *Europhys. Lett.* **36** 209
[7] Halpin-Healy T and Zhang Y-C 1995 *Phys. Rep.* **254** 215
[8] Parisi G 1990 *J. Physique* **51** 1595
[9] Mezard M 1990 *J. Physique* **51** 1831
[10] Ludwig W and Falter C 1988 *Symmetries in Physics* (Berlin: Springer)
[11] Hamermesh M 1964 *Group Theory and its Application to Physical Problems* (Reading, MA: Addison Wesley)
[12] Derrida B and Hilhorst H J 1983 *J. Phys. A: Math. Gen.* **16** 2641
[13] Lin T F 1970 *J. Math. Phys.* **11** 1584
[14] Binder K and Young A P 1986 *Rev. Mod. Phys.* **58** 801